

SUMS OF SETS OF LATTICE POINTS AND UNIMODULAR COVERINGS OF POLYTOPES

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ABSTRACT. If P is a lattice polytope (that is, the convex hull of a finite set of lattice points in \mathbf{R}^n), then every sum of h lattice points in P is a lattice point in the h -fold sumset hP . However, a lattice point in the h -fold sumset hP is not necessarily the sum of h lattice points in P . It is proved that if the polytope P is a union of unimodular simplices, then every lattice point in the h -fold sumset hP is the sum of h lattice points in P .

1. THE ADDITION PROBLEM FOR LATTICE POLYTOPES AND POLYHEDRA

The *sumset*, also called the *Minkowski sum*, of sets X_1, \dots, X_h in \mathbf{R}^n is the set

$$X_1 + \dots + X_h = \{x_1 + \dots + x_h : x_i \in X_i \text{ for all } i = 1, \dots, h\}.$$

If $x_i \in X_i \cap \mathbf{Z}^n$ for $i = 1, \dots, h$, then $x_1 + \dots + x_h \in \mathbf{Z}^n$, and so

$$(1) \quad (X_1 \cap \mathbf{Z}^n) + \dots + (X_h \cap \mathbf{Z}^n) \subseteq (X_1 + \dots + X_h) \cap \mathbf{Z}^n.$$

It is an unsolved problem in additive number theory to describe the h -tuples of sets X_1, \dots, X_h in \mathbf{R}^n for which equality replaces inclusion in the relation (1). In particular, this is an unsolved problem in convex geometry, even in the important case when the sets X_1, \dots, X_h are lattice polytopes.

For every positive integer h and every $X \subseteq \mathbf{R}^n$, we define the *h -fold sumset*

$$hX = \underbrace{X + \dots + X}_{h \text{ summands}} = \{x_1 + \dots + x_h : x_i \in X \text{ for all } i = 1, \dots, h\}$$

and, for every positive real number λ , we define the *dilation*

$$\lambda * X = \{\lambda x : x \in X\}.$$

If X is convex, then $hX = h * X$, that is, the h -fold sumset equals the dilation by h .

It is also an unsolved problem to determine necessary and sufficient conditions for a lattice polytope P to satisfy the equation

$$(2) \quad h(P \cap \mathbf{Z}^n) = (hP) \cap \mathbf{Z}^n$$

for some integer $h \geq 2$, or for all $h \geq 2$, or for all sufficiently large h .

In this paper we give a simple sufficient condition for a lattice polytope P to satisfy equation (2) for all positive integers h , and we show that this sufficient

Date: November 13, 2015.

2010 Mathematics Subject Classification. Primary 11B13, 11P21, 52A10, 52B20, 52C05.

Key words and phrases. Sums of sets of lattice points, lattice polytope, unimodular simplex, unimodular covering.

Supported in part by a grant from the PSC-CUNY Research Award Program.

condition implies that for every lattice polytope P there is a positive integer ℓ such that the lattice polytope ℓP satisfies

$$(3) \quad h(\ell P \cap \mathbf{Z}^n) = (h\ell P) \cap \mathbf{Z}^n$$

all positive integers h .

The sufficient condition is that the lattice polytope P have a unimodular cover. Both the addition problem for lattice points in polytopes and unimodular covers and triangulations of polytopes have been extensively investigated. For the addition problem, see [5, 7, 9, 10]. For unimodular covers, see [1, 2, 3, 4, 6, 8, 11, 12].

2. UNIMODULAR SIMPLICES

Let $A = \{a_0, a_1, \dots, a_n\}$ be an affinely independent set in \mathbf{R}^n . The n -dimensional simplex generated by A is the convex hull of A , that is, the set

$$\begin{aligned} \Delta(A) &= \left\{ \sum_{i=0}^n t_i a_i : t_i \geq 0 \text{ for } i = 0, 1, \dots, n \text{ and } \sum_{i=0}^n t_i = 1 \right\} \\ &= a_0 + \left\{ \sum_{i=1}^n t_i (a_i - a_0) : t_i \geq 0 \text{ for } i = 1, \dots, n \text{ and } \sum_{i=1}^n t_i \leq 1 \right\}. \end{aligned}$$

A *lattice simplex* is an n -dimensional simplex $\Delta(A)$, where A is a set of $n+1$ affinely independent lattice points.

Let $\Delta(A)$ be a lattice simplex, and let $\Gamma(A)$ be the subgroup of \mathbf{Z}^n generated by $A - A$. The simplex $\Delta(A)$ is *unimodular* if $\Gamma(A) = \mathbf{Z}^n$.

For example, the *standard simplex* in \mathbf{R}^n is the lattice simplex $\Delta = \Delta(\{0\} \cup \mathcal{E})$, where $\mathcal{E} = \{e_1, \dots, e_n\}$ is the standard basis for \mathbf{R}^n . This simplex is unimodular. In \mathbf{R}^3 , the simplex generated by the set

$$A_1 = \{0, e_1, e_2, 2e_3\}$$

is not unimodular, because

$$\Gamma(A_1) = \{(x_1, x_2, x_3) \in \mathbf{Z}^3 : x_3 \equiv 0 \pmod{2}\}$$

is a subgroup of \mathbf{Z}^3 of index 2. Note that $\Delta(A_1) \cap \mathbf{Z}^n = A_1 \cup \{e_3\}$.

The simplex generated by the set

$$A_2 = \{0, e_1, e_2, e_1 + e_2 + 2e_3\}$$

satisfies $\Delta(A_2) \cap \mathbf{Z}^n = A_2$, but $\Gamma(A_1) = \Gamma(A_2)$, and so $\Delta(A_2)$ is not unimodular.

Lemma 1. *Let $A = \{a_0, a_1, \dots, a_n\}$ be an affinely independent set in \mathbf{Z}^n . If the simplex $\Delta(A)$ is unimodular, then*

$$hA = h\Delta(A) \cap \mathbf{Z}^n$$

every positive integer h .

Proof. We have $A \subseteq \Delta(A)$ and so $hA \subseteq h\Delta(A)$. Because $hA \subseteq \mathbf{Z}^n$, it follows that $hA \subseteq h\Delta(A) \cap \mathbf{Z}^n$.

Conversely, let $p \in h\Delta(A) \cap \mathbf{Z}^n$. There exist nonnegative real numbers t_0, t_1, \dots, t_n such that

$$\sum_{i=0}^n t_i = h$$

and

$$p = \sum_{i=0}^n t_i a_i = \left(h - \sum_{i=1}^n t_i \right) a_0 + \sum_{i=1}^n t_i a_i = h a_0 + \sum_{i=1}^n t_i (a_i - a_0) \in \mathbf{Z}^n.$$

It follows that

$$(4) \quad p - h a_0 = \sum_{i=1}^n t_i (a_i - a_0).$$

The affine independence of the set A implies that the set $A - A = \{a_i - a_0 : i = 1, \dots, n\}$ is an \mathbf{R} -basis for \mathbf{R}^n . The unimodality of $\Delta(A)$ implies that the set $\{a_i - a_0 : i = 1, \dots, n\}$ is a \mathbf{Z} -basis for the free abelian group \mathbf{Z}^n , and so there exist integers w_1, \dots, w_n such that

$$(5) \quad p - h a_0 = \sum_{i=1}^n w_i (a_i - a_0).$$

Comparing equations (4) and (5), we see that

$$t_i = w_i \in \mathbf{N}_0$$

for $i = 1, \dots, n$. It follows that

$$w_0 = h - \sum_{i=1}^n w_i = h - \sum_{i=1}^n t_i = t_0 \in \mathbf{N}_0.$$

We have

$$\sum_{i=0}^n w_i = \sum_{i=0}^n t_i = h$$

and so

$$p = \sum_{i=0}^n w_i a_i \in hA.$$

This completes the proof. \square

A similar argument proves that if $\Delta(A)$ is a unimodular simplex in \mathbf{R}^n , then $A = \Delta(A) \cap \mathbf{Z}^n$.

Theorem 1. *If P is a lattice polytope that is the union of unimodular simplices, then*

$$h(P \cap \mathbf{Z}^n) = (hP) \cap \mathbf{Z}^n$$

for every positive integer h . Moreover, if $x \in (hP) \cap \mathbf{Z}^n$, then $P \cap \mathbf{Z}^n$ contains an affinely independent set $A = \{a_0, a_1, \dots, a_n\}$ such that $x = \sum_{i=0}^n t_i a_i$ with $t_i \in \mathbf{N}_0$ for $i = 0, 1, \dots, n$ and $\sum_{i=0}^n t_i = h$.

Proof. If $p \in (hP) \cap \mathbf{Z}^n$, then $(1/h)p \in P$ and there is a unimodular simplex $\Delta(A) \subseteq P$ such that $(1/h)p \in \Delta(A)$. It follows that $p \in h\Delta(A) \cap \mathbf{Z}^n$. By Lemma 1,

$$p \in h\Delta(A) \cap \mathbf{Z}^n = hA \subseteq h(P \cap \mathbf{Z}^n).$$

This completes the proof. \square

Theorem 2. *For every lattice polytope P there is a positive integer ℓ such that equation*

$$h(\ell P \cap \mathbf{Z}^n) = (h\ell P) \cap \mathbf{Z}^n$$

holds for every positive integer h that is a multiple of ℓ .

Proof. Every lattice polytope P has a dilation that has a unimodular cover (Bruns and Gubeladze [3, Chapter 3]). If ℓP has a unimodular cover, then ℓP satisfies equation (3). This completes the proof. \square

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